Odd 2–factored snarks

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Abstract

A *snark* is a cubic cyclically 4–edge connected graph with edge chromatic number four and girth at least five. We say that a graph G is *odd 2–factored* if for each 2–factor F of G each cycle of F is odd.

In this paper, we present a method for constructing odd 2–factored snarks. In particular, we construct two families of odd 2–factored snarks that disprove a conjecture by some of the authors. Moreover, we approach the problem of characterizing odd 2–factored snarks furnishing a partial characterization of cyclically 4–edge connected odd 2–factored snarks. Finally, we pose a new conjecture regarding odd 2–factored snarks.

1   Introduction

All graphs considered are finite and simple (without loops or multiple edges). We shall use the term multigraph when multiple edges are permitted. For definitions and notations not explicitly stated the reader may refer to [10].

A *snark* (cf. e.g. [24]) is a bridgeless cubic graph with edge chromatic number four (by Vizing's theorem the edge chromatic number of every cubic graph is either three or four so a snark corresponds to the special case of four). In order to avoid trivial cases, snarks are usually assumed to have girth at least five and not to contain a non–trivial 3–edge cut (i.e. they are cyclically 4–edge connected).

Snarks were named after the mysterious and elusive creature in Lewis Caroll's famous poem *The Hunting of The Snark* by Martin Gardner in 1976 [20], but it was P. G. Tait in 1880 that initiated the study of snarks, when he proved that the four colour theorem is equivalent to the statement that *no snark is planar* [34]. The Petersen graph P is the smallest snark and Tutte conjectured that all snarks have Petersen graph minors. This conjecture was confirmed by Robertson, Seymour and Thomas (unpublished, see [31]). Necessarily, snarks are non–hamiltonian.

The importance of the snarks does not only depend on the four colour theorem. Indeed, there are several important open problems such as the classical cycle double cover conjecture [32, 33], Fulkerson's conjecture [16] and Tutte's 5–flow conjecture [35] for which it is sufficient to prove them for snarks. Thus, minimal counterexamples to these and other problems must reside, if they exist at all, among the family of snarks.

Snarks play also an important role in characterizing regular graphs with some conditions imposed on their 2–factors. Recall that a 2–factor is a 2–regular spanning subgraph of a graph G.

A graph with a 2–factor is said to be 2–*factor hamiltonian* if all its 2–factors are Hamilton cycles, and, more generally, 2–*factor isomorphic* if all its 2–factors are isomorphic. Examples of such graphs are K_4, K_5, $K_{3,3}$, the Heawood graph (which are all 2–factor hamiltonian) and the Petersen graph (which is 2–factor isomorphic). Moreover, a *pseudo* 2–*factor isomorphic graph* is a graphs G with the property that the parity of the number of cycles in a 2–factor is the same for all 2–factors of G. Example of these graphs are $K_{3,3}$, the Heawood graph H_0 and the Pappus graph P_0. Several papers have addressed the problem of characterizing families of graphs (particularly regular graphs) which have these properties directly [19, 6, 1, 2, 3, 4, 5, 12, 11] or indirectly [17, 27, 28, 29, 18, 7, 15]. In particular, we have recently pointed out in [4] some relations between snarks and some of these families (cf. Section 2).

We say that a graph G is *odd* 2–*factored* (cf. [4]) if for each 2–factor F of G each cycle of F is odd. In [4] we have investigated which snarks are odd

2–factored and we have conjectured that *a snark is odd* 2–*factored if and only if G is the Petersen graph, Blanuša* 2*, or a Flower snark $J(t)$, with $t \geq 5$ and odd* (Conjecture 2.5).

At present, there is no uniform theoretical method for studying snarks and their behaviour. In particular, little is known about the structure of 2–factors in a given snark.

In this paper, we present a new method, called *bold–gadget dot product*, for constructing odd 2–factored snarks using the concepts of bold–edges and gadget–pairs over Isaacs' dot–product [25]. This method allows us to construct two new instances of odd 2–factored snarks of order 26 and 34 that disprove the above conjecture (cf. Conjecture 2.5). Moreover, we furnish a characterization of bold–edges and gadget–pairs in known odd 2–factored snarks and we approach the problem of characterizing odd 2–factored snarks furnishing a partial characterization of cyclically 4–edge connected odd 2–factored snarks. Finally, we pose a new conjecture about odd 2–factored snarks.

2   Preliminaries

Until 1975 only five snarks were known, then Isaacs [25] constructed two infinite families of snarks, one of which is the Flower snark [25], for which in [4] we have used the following definition:

Let $t \geq 5$ be an odd integer. The *Flower snark* (cf. [25]) $J(t)$ is defined in much the same way as the graph $A(t)$ described in [1].

The graph $J(t)$ has vertex set

$$V(t) = \{h_i,\, u_i,\, v_i,\, w_i \,:\, i = 1, 2, \ldots, t\}$$

and edge set

$$\begin{aligned} E(t) = \{&h_i u_i,\, h_i v_i,\, h_i w_i,\, u_i u_{i+1},\, v_i v_{i+1},\, w_i w_{i+1},\, :\, i = 1, 2, \ldots, t-1\} \\ &\cup \{u_t v_1,\, v_t u_1,\, w_1 w_t\} \end{aligned}$$

For $i = 1, 2, \ldots, t$ we call the subgraph IC_i of $J(t)$ induced by the vertices $\{h_i,\, u_i,\, v_i,\, w_i\}$ the ith *interchange* of $J(t)$. The vertices h_i and the edges $\{h_i u_i,\, h_i v_i,\, h_i w_i\}$ are called respectively the *hub* and the *spokes* of IC_i. The set of edges $\{u_i u_{i+1},\, v_i v_{i+1},\, w_i w_{i+1}\}$ linking IC_i to IC_{i+1} are said to be the ith *link L_i* of $J(t)$. The edge $u_i u_{i+1} \in L_i$ is called the *u–channel of the link*. The subgraph of $J(t)$ induced by the vertices $\{u_i,\, v_i \,:\, i = 1, 2, \ldots, t\}$ and $\{w_i \,:\, i = 1, 2, \ldots, t\}$ are respectively cycles of length $2t$ and t and are said to be the *base cycles* of $J(t)$.

The technique used by Isaacs to construct the second infinite family is called *dot product* and it is a consequence of the following:

Lemma 2.1 (*Parity Lemma*) *[25, 36] Let G be a cubic graph and let $c : E(G) \to \{1, 2, 3\}$ a 3–edge–coloring of G. Then, for every edge cut T in G,*

$$|T \cap c^{-1}(i)| \equiv |T| \, mod \, 2$$

for each $i \in \{1, 2, 3\}$.

A *dot product* of two connected cubic graphs L and R denoted by $G = L \cdot R$ is defined as follows [25, 24]:

1. remove any pair of adjacent vertices x and y from L;

2. remove any two independent edges ab and cd from R;

3. join $\{r, s\}$ to $\{a, b\}$ and $\{t, u\}$ to $\{c, d\}$ or $\{r, s\}$ to $\{c, d\}$ and $\{t, u\}$ to $\{a, b\}$, where $N(x) - y = \{r, s\}$ and $N(y) - x = \{t, u\}$.

$$L \qquad\qquad R \qquad\qquad\qquad\qquad L \cdot R$$
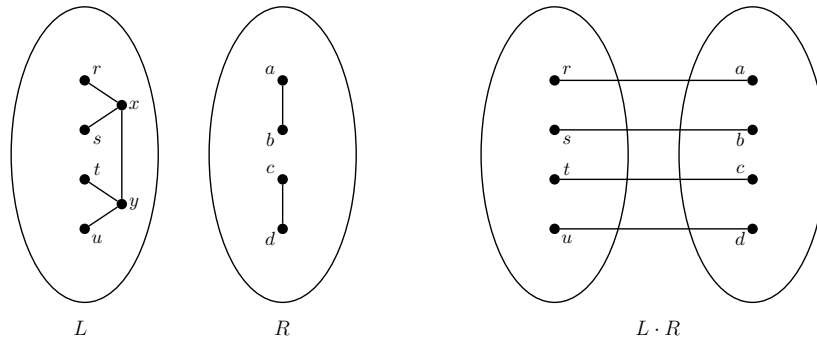
Note that the dot product allows to construct graphs of cyclic edge–connectivity exactly 4. Moreover, the dot product of the Petersen graph with itself $P \cdot P$ gives rise to two snarks Blanuša 1 and Blanuša 2.

The Parity Lemma 2.1 allows to prove the following:

Theorem 2.2 *[25, 36] Let L and R be snarks. Then the dot product $L \cdot R$ is also a snark.*

A more general method to construct snarks called *superposition* has been introduced by M. Kochol [26]. A superposition is performed replacing simultaneously edges and vertices of a snark by suitable cubic graphs with pendant (or half) edges (called superedges and supervertices) yielding a new

snark. Superpositions allow to construct cyclically k–edge–connected snarks with arbitrarily large girth, for $k = 4, 5, 6$.

As already mentioned in the Introduction a graph G is *odd 2–factored* if for each 2–factor F of G each cycle of F is odd.

By definition, *an odd 2–factored graph G is pseudo 2–factor isomorphic.* Note that, odd 2–factoredness is not the same as the *oddness* of a (cubic) graph (cf. e.g.[36]).

Lemma 2.3 *[4] Let G be a cubic 3–connected odd 2–factored graph then G is a snark.*

In [4] we have posed the following:

QUESTION: *Which snarks are odd 2–factored?*

and we have proved:

Proposition 2.4 *[4]*

(i) *Petersen and Blanuša2 are odd 2–factored snarks.*

(ii) *The Flower Snark $J(t)$, for odd $t \geq 5$, is odd 2–factored. Moreover, $J(t)$ is pseudo 2–factor isomorphic but not 2–factor isomorphic.*

(iii) *All other known snarks up to 22 vertices and all other named snarks up to 50 vertices are not odd 2–factored.*

Thus it seemed reasonable to pose the following:

Conjecture 2.5 *[4] A snark is odd 2–factored if and only if G is the Petersen graph, Blanuša 2, or a Flower snark $J(t)$, with $t \geq 5$ and odd.*

We disprove Conjecture 2.5 in Section 4.

As mentioned above, the Blanuša graphs arise as the dot product of the Petersen graph with itself, but one is odd 2–factored (cf. Proposition 2.4(i)) while the other one is not. In the Petersen graph, which is edge transitive, there are exactly two kinds of pairs of independent edges. The Blanuša snarks are the result of these two different choices of the independent edges in the dot product. We will make use of this property for constructing new odd 2–factored snarks in Sections 3 and 4.

Proposition 2.6 *The dot product preserves snarks, but not odd 2-factored graphs.*

Proof.   It is immediate from Theorem 2.2 and Proposition 2.4(i), (iii).   □

3   A construction of odd 2–factored snarks

We present a general construction of odd 2–factored snarks performing the dot product on edges with particular properties, called *bold–edges* and *gadget–pairs* respectively, of two snarks L and R.

CONSTRUCTION: BOLD–GADGET DOT PRODUCT.

We construct (new) odd 2–factored snarks as follows:

- Take two snarks L and R with bold–edges (cf. Definition 3.1) and gadget–pairs (cf. Definition 3.3), respectively;

- Choose a bold–edge xy in L;

- Choose a gadget–pair f, g in R;

- Perform the dot product $L \cdot R$ using these edges;

- Obtain a new odd 2–factored snark (cf. Theorem 3.7).

Note that in what follows the existence of a 2–factor in a snark is guaranteed since they are bridgeless by definition.

Definition 3.1 *Let L be a snark. A* bold–edge *is an edge $e = xy \in L$ such that the following conditions hold:*

(i) *All 2–factors of $L - x$ and of $L - y$ are odd;*

(ii) *all 2–factors of L containing xy are odd;*

(iii) *all 2–factors of L avoiding xy are odd.*

Note that not all snarks contain bold–edges (cf. Proposition 4.2, Lemma 5.1). Furthermore, conditions (ii) and (iii) are trivially satisfied if L is odd 2–factored.

Lemma 3.2 *The edges of the Petersen graph P_{10} are all bold–edges.*

Proof. Since P_{10} is hypohamiltonian (i.e. $P_{10} - v$ is hamiltonian, for each $v \in V(P_{10})$) and moreover, for every $v \in P_{10}$, all 2–factors of $P_{10} - v$ are hamiltonian , condition (i) holds. The other two conditions are satisfied since P_{10} is odd-2–factored. □

Definition 3.3 *Let R be a snark. A pair of independent edges $f = ab$ and $g = cd$ is called a* gadget–pair *if the following conditions hold:*

(i) *There are no 2–factors of R avoiding both f, g;*

(ii) *all 2–factors of R containing exactly one out of $\{f, g\}$ are odd;*

(iii) *all 2–factors of R containing both f and g are odd. Moreover, f and g are contained in different cycles.*

(iv) *all 2–factors of $R - \{f, g\} \cup \{ac, ad, bc, bd\}$ containing exactly one edge out of $\{ac, ad, bc, bd\}$, are such that the cycle containing the new edge is even and all other cycles are odd.*

Note that, finding gadget–pairs in a snark is not an easy task and, in general, not all snarks contain gadget–pairs (cf. Lemma 5.2).

Let $H := \{x_1y_1, x_2y_2, x_3y_3\}$ be the two horizontal edges and the vertical edge respectively (in the pentagon–pentagram representation) of P_{10} (cf. Figure 1).
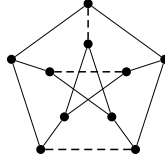

Figure 1: Any pair of the dashed edges is a gadget–pair in P_{10}

It is easy to prove the following properties:

Lemma 3.4 *Let P_{10} be the Petersen graph and $H := \{x_1y_1, x_2y_2, x_3y_3\}$ be as above.*

(i) *The graph $P_{10} - H$ is bipartite.*

(ii) *The graph $P_{10} - \{f, g\}$ has no 2–factors, for any distinct $f, g \in H$.*

Lemma 3.5 *Any pair of edges f, g in the set H of P_{10} is a gadget–pair.*

Proof. It can be easily checked that the edges of H in P_{10} have the property that any 2–factor of P_{10} contains exactly two of them. Moreover, they belong to different cycles of the 2–factor. Indeed, for any two edges f and g in H their endvertices are all at distance 2 in P_{10}. Thus the shortest cycle containing both f and g has length 6. Since all 2–factors of P_{10} contain two 5–cycles, in any 2–factor of P_{10} containing both f and g, these edges are contained in different cycles. Hence, conditions (i)–(iii) follow from the above reasoning and the odd 2–factoredness of P_{10}.

Condition (iv) can also be easily checked and moreover, any 2–factor of $P_{10} - \{f, g\} + \{x_ix_j\}$ (or $P_{10} - \{f, g\} + \{x_iy_j\}$ or $P_{10} - \{f, g\} + \{y_iy_j\}$), for

$i \neq j$, containing the new edge is hamiltonian, hence even (and obviously there are no other cycles in these 2–factors). □

In the next lemma we recall a well known property of edge–cuts:

Lemma 3.6 *Let G be a connected graph and let S be a set of edges such that $G - S$ is disconnected, but $G - S'$ is not disconnected, for any proper subset S' of S. Then, for any cycle C of G, $E(C) \cap S$ is even.*

The following theorem allows us to construct new odd 2–factored snarks.

Theorem 3.7 *Let xy be a bold–edge in a snark L and let $\{ab, cd\}$ be a gadget– pair in a snark R. Then $L \cdot R$ is an odd 2–factored snark.*

Proof.   Denote $e := xy$, $f := ab$, $g := cd$, $N(x) - y := \{r, s\}$, $N(y) - x := \{t, u\}$ and $T := \{ra, sb, tc, ud\}$ the 4–edge cut obtained performing the dot product $L \cdot R$.

Let F be a 2–factor of $L \cdot R$ then F contains an even number of edges of T by Lemma 3.6.

We distinguish three cases according to the number of edges of T in F:

CASE 1. F contains no edges of T.

In this case it is immediate to check that a subset of the cycles of F forms a 2–factor of $R - \{f, g\}$, contradicting Definition 3.3(i). Thus there are no 2–factors of $L \cdot R$ avoiding T.

CASE 2. F contains exactly two edges e_1 and e_2 of T.

We want to prove that all cycles of F are odd. We distinguish two subcases.
CASE 2.1. The endvertices of e_1 and e_2 in R are both endvertices of either f or g.

W.l.g. we may assume that $e_1 = ra$ and $e_2 = sb$. Let $F_1 := F \cap L$ and let $F_1' := F_1 \cup \{rx, sx\}$. Then F_1' is a 2–factor of $L - y$. Analogously, let $F_2 := F \cap R$ and let $F_2' := F_2 \cup \{f\}$. Then F_2' is a 2–factor of R containing f and avoiding g. Let C_x be the cycle of F_1' containing x. Then $|C_x|$ is odd by Definition 3.1(i). Similarly, let C_f be the cycle of F_2' containing f. Then $|C_f|$ is odd by Definition 3.3(ii). Thus, the cycle C of F containing e_1 and e_2 has $|C| = |C_x| - 2 + |C_f| - 1 + 2 = |C_x| + |C_f| - 1$ which is odd. Finally,all other cycles of F are odd by Definition 3.1(i) and Definition 3.3(ii).
CASE 2.2. The endvertices of e_1 and e_2 in R lie one in f and the other in g.

W.l.g. we may assume that $e_1 = ra$ and $e_2 = tc$. Let $F_1 := F \cap L$ and let $F_1' := F_1 \cup \{rx, xy, yt\}$. Then F_1' is a 2–factor of L containing xy. Analogously, let $F_2 := F \cap R$ and let $F_2' := F_2 \cup \{ac\}$. Let $S := \{ac, ad, bc, bd\}$ be a set of new edges and consider the graph $R' := R - \{f, g\} \cup S$. Then F_2' is a 2–factor of R'

containing only ac of S by construction. Let C_{xy} be the cycle of F_1' containing xy. Then $|C_{xy}|$ is odd by Definition 3.1(ii). Similarly, let C_{ac} be the cycle of F_2' containing ac. Then $|C_{ac}|$ is even by Definition 3.3(iv). Thus, the cycle C of F containing e_1 and e_2 has $|C| = |C_{xy}| - 3 + |C_{ac}| - 1 + 2 = |C_{xy}| + |C_{ac}| - 2$ which is odd. Finally, all other cycles of F are odd by Definition 3.1(ii) and Definition 3.3(iv).

CASE 3. F contains all the four edges of T.

Again we want to prove that all cycles of F have odd length. Let $F_1 := F \cap L$, $F_2 := F \cap R$, $F_1' := F_1 \cup \{rx, sx, ty, uy\}$ and $F_2' := F_2 \cup \{ab, cd\}$. Note that F_1' is a 2–factor of L avoiding xy and that F_2' is a 2–factor of R containing both f and g. Let C_x and C_y be the cycles of F_1' containing x and y, respectively. If $C_x = C_y$ then we denote such cycle by C_{xy}. Analogously, let C_f and C_g be the cycles of F_2' containing f and g, respectively. Note that C_f and C_g are always distinct by Definition 3.3(iii).

In order to compute the parity of the length of the cycles of F containing T, We need to analyze all possible combinations of paths in F between the vertices $\{r, s, t, u\}$ and between the vertices $\{a, b, c, d\}$ of $L \cdot R$. It is easy to check that we have five different cases (the others being equivalent to some of these five) but three of them are ruled out by Definition 3.3(iii), since they have $C_f = C_g$ (cf. Figure 2).

CASE 3.1   CASE 3.2
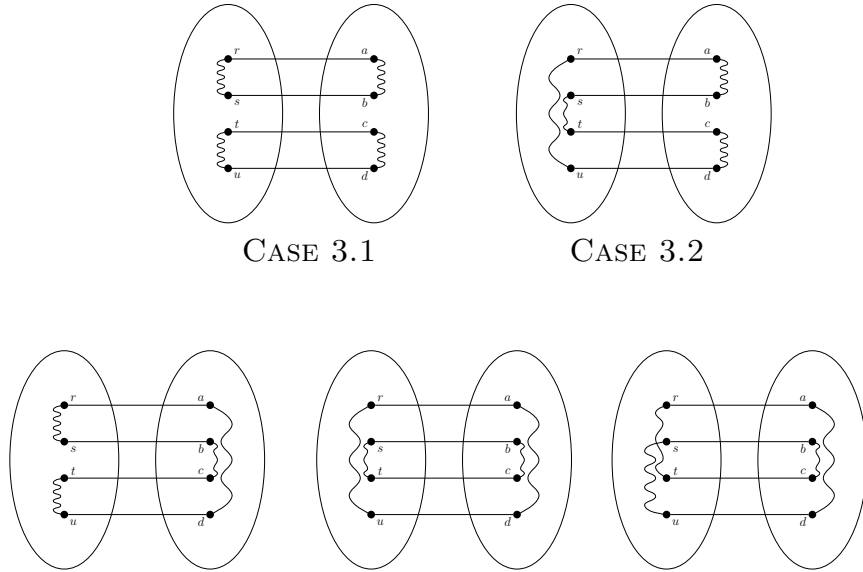
These three cases can be ruled out since they give rise to
$C_f = C_g$ in F_2', impossibile by Definition 3.3(iii)

Figure 2: Subcases of CASE 3

The two remaining subcases are:

CASE 3.1. All edges of T lie in a cycle C of F such that $C_x = C_y = C_{xy}$ and $C_f \neq C_g$.

In this case $|C| = |C_{xy}| - 4 + |C_f| - 1 + |C_g| - 1 + 4 = |C_{xy}| + |C_f| + |C_g| - 2$ which is odd by Definition 3.1(iii) and Definition 3.3(iii).

CASE 3.2. The edges of T are contained in two distinct cycles C_1 and C_2 of F such that $C_x \neq C_y$ and $C_f \neq C_g$.

In this case $|C_1| = |C_x| - 2 + |C_f| - 1 + 2 = |C_x| + |C_f| - 1$ and $|C_2| = |C_y| - 2 + |C_g| - 1 + 2 = |C_y| + |C_g| - 1$ which are odd by Definition 3.1(iii) and Definition 3.3(iii).

All the remaining cycles of F are odd by Definition 3.1(iii) and Definition 3.3(iii).

Thus, the resulting graph is odd 2–factored, hence a snark by Lemma 2.3. □

CONSTRUCTION OF P_{18}

Recall that, the Blanuša2 snark is odd 2–factored (cf. [4] and Proposition 2.4). We can obtain the same result taking two copies L, R of the Petersen graph P_{10}, the first one choosing any edge as a bold–edge (by Lemma 3.2) and the second with a gadget–pair as in Lemma 3.5. The resulting graph, obtained as the dot product $L \cdot R$, denoted by P_{18}, is odd 2–factored by Theorem 3.7 (cf. Figure 4) and it is isomorphic to the Blanuša2 snark.

Let $H := \{e_1, e_2, e_3\}$ be the two horizontal edges and the vertical edge respectively (in the pentagon–pentagram representation) of the Petersen graph P_{10}, as in Figure 1. Let L and R be two copied of P_{10}. Choose $e_1 = xy$ as the bold–edge in L and $f, g \in H$ as the gadget–pair in R. Moreover, let $L_0 := L - \{x, y\}$ and $R_0 := R - \{f, g\}$ be the 4–*poles* represented as follows:
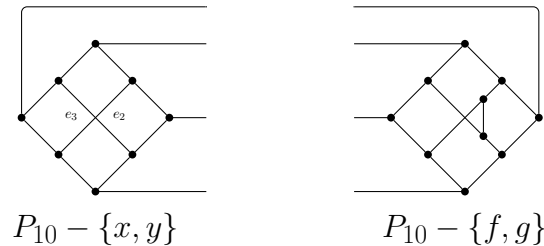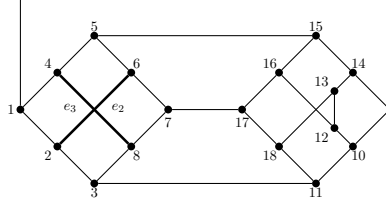

$$P_{10} - \{x, y\} \qquad\qquad P_{10} - \{f, g\}$$

Figure 3: 4–poles from Petersen

Performing the dot–product $L \cdot R$ we obtain the Blanuša2 snark P_{18} (Figure 4).

Figure 4: P_{18}

Lemma 3.8 *Under the above hypothesis, the only bold–edges of P_{18} are those edges, say e_2 and e_3, identified with the homonymous edges of L (cf. Figure 3.8).*

Proof. Fix the labelling on $V(P_{18})$ as in Figure 4. To find all possible bold–edges in P_{18}, we only need to verify Definition 3.1(i), since P_{18} is odd 2–factored.

   To this purpose, we have implemented a program, with the software package MAGMA [8], and computed that the graph P_{18} has the dihedral group D_4 as automorphism group, its edge–orbits are six and its vertex–orbits are five. For each representative v of the vertex–orbits, we have determined all the 2–factors of $P_{18} - v$ (computing the determinant of the variable adjacency matrix of G [23]). The only vertex, for which $P_{18} - v$ has only odd 2–factors, is $v = 2$ (lying in the vertex–orbit $\{2, 4, 6, 8\}$). Hence, the only bold–edges in P_{18} are e_2, e_3, since there is an edge–orbit $E_0 := \{(2, 6), (4, 8)\}$ of P_{18}, and its edges correspond to e_2, e_3 (c.f. Figure 4). $\square$

4   Counterexamples to Conjecture 2.5: Constructions

We construct two new example of odd 2–factored snarks of order 26 and 34, called P_{26} and P_{34}, starting from the Petersen and the Blanuša2 snarks applying iteratively the method described in Section 3. These two examples disprove Conjecture 2.5. Moreover, we investigate the structure of the snarks obtained with this method computing their bold–edges and gadget pairs.

CONSTRUCTION OF P_{26}

Proposition 4.1 *Let L be a copy of P_{18} and R be a copy of P_{10}. Choose $e_2 = xy$ to be one of the two bold–edges in L and let $f, g \in H$ be a gadget–pair*

in R. Then the dot product $L \cdot R$ gives rise to a new odd 2–factored snark P_{26}. Moreover, the only bold–edge of P_{26} is e_3, the edge of P_{26} identified with the homonymous edge of P_{18} (cf. Figure 5).
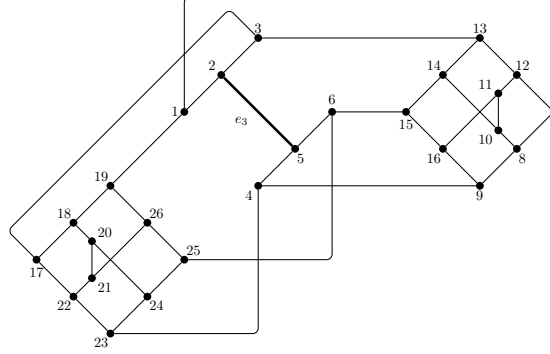

Figure 5: Labels for P_{26}

Proof.   Applying the construction given by Theorem 3.7 to the chosen bold–edge $e_2 \in L$ (cf. Lemma 3.8) and gadget–pair $f, g \in R$ (cf. Lemma 3.5), we obtain that the graph P_{26} is an odd 2–factored snark.
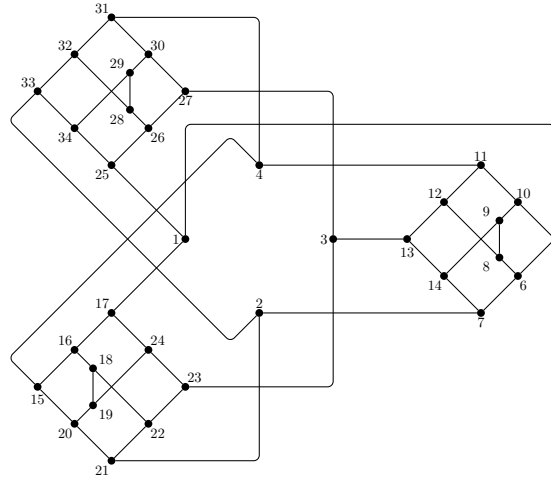
Fix the labelling on $V(P_{26})$ as in Figure 5. To find all possible bold–edges in P_{26}, we only need to verify Definition 3.1(i), since we have just proved that P_{26} is odd 2–factored.

To this purpose, as in Proof of Lemma 3.8, we have implemented a program, with the software package MAGMA, and computed that the graph P_{26} has the dihedral group D_4 as automorphism group, its edge–orbits are eight and its vertex–orbits are seven. For each representative v of the vertex–orbits, we have determined all the 2–factors of $P_{26} - v$. The only vertex, for which $P_{26} - v$ has only odd 2–factors, is $v = 2$ (lying in the vertex–orbit $\{2, 5\}$). Hence, the only bold–edge in P_{26} is e_3, since there is an edge–orbit $E_0 := \{(2, 5)\}$ of P_{26}, and its edge correspond to e_3 (c.f. Figure 5).   □

CONSTRUCTION OF P_{34}

Proposition 4.2 *Let L be a copy of P_{26} and R be a copy of P_{10}. Choose $e_3 = xy$ to be the only bold–edge in L and let $f, g \in H$ be a gadget–pair in R. Then the dot product $L \cdot R$ gives rise to a new odd 2–factored snark P_{34}. Moreover, P_{34} has no bold–edges (cf. Figure 6).*

Proof.   Applying the construction given by Theorem 3.7 to the only bold–edge $e_3 \in L$ (cf. Proposition 4.1) and gadget–pair $f, g \in R$ (cf. Lemma 3.5), we obtain that the graph P_{34} is an odd 2–factored snark.

Figure 6: Labels for P_{34}

Fix the labelling on $V(P_{34})$ as in Figure 6. To find all possible bold–edges in P_{34}, again, we only need to verify that Definition 3.1(i) does not hold, since we have just proved that P_{34} is odd 2–factored.

To this purpose, as in Lemma 3.8 and Proposition 4.1, we have implemented a program, with the software package MAGMA, and computed that the graph P_{34} has the symmetric group S_4 as automorphism group, its edge–orbits and its vertex–orbits are both four. For each representative v of the vertex–orbits, we have determined all the 2–factors of $P_{34} - v$. We have obtained that there is always a 2–factor containing a cycle of even length. Thus, Definition 3.1(i) does not hold. Hence P_{34} has no bold–edges.   □

Remark 4.3 *We have learned from J. Hägglund [21] that Brimnkmann, Goedbgebeur, Markstrom and himself had also found in [9], with an exhaustive computer search of all snarks of order $n \leq 36$, numerical counterexamples to Conjecture 2.5, one of order 26 and one of order 34, but at the time we have informed him that we had already constructed these counterexamples via the bold–gadget dot product. Indeed, we have checked that the snarks P_{26} and P_{34} are isomorphic to their graphs of order 26 and 34, respectively.*

5 A partial characterization of odd 2–factored snarks

To approach the problem of characterizing all odd 2–factored snarks, we consider the possibility of constructing further odd 2–factored snarks with the technique presented in Section 3, which relies in finding other snarks with

bold–edges and/or gadget–pairs, Therefore, we study the existence of bold–edges and gadget–pairs in the already constructed odd 2–factored snarks.

We have already computed all the bold–edges in the Petersen graph P_{10}, the Blanuša2 snark P_{18}, the new snarks P_{26} and P_{34} (cf. Lemma 3.2, Lemma 3.8, Proposition 4.1, Proposition 4.2).

Lemma 5.1 *Let $J(t)$, for odd $t \geq 5$, be the Flower Snark. Then $J(t)$ has no bold–edges.*

Proof. Fix the labelling on the vertices of $J(t)$ as defined in Section 2. The flower snark has the dihedral group D_{2t} as automorphism group [13], its edge–orbits are four and its vertex–orbits are three.

To prove that there are no bold–edges, we only need to verify Definition 3.1(i) does not hold, since we have already proved in [4] that $J(t)$ is odd 2–factored. To this purpose, we have to find a 2–factor containing an even cycle in $J(t) - v$, for each representative v of the vertex–orbits.

Let h_1, w_1 and u_1 be representatives for the three vertex–orbits of $J(t)$. Then, for each orbit we can construct the following 2–factor in $J(t) - v$:

Graph	2–factor cycles	lengths
$J(t) - h_1$	$(w_i, h_i, v_i, v_{i+1}, h_{i+1}, w_{i+1})$ for $i = 3, \ldots, t-2$	$\frac{t-3}{2}$ cycles of length 6
	$(u_1, u_2, \ldots, u_t, v_1, v_2, h_2, w_2, w_1, w_t, h_t, v_t)$	a cycle of length $t+8$
$J(t) - w_1$	$(w_i, h_i, v_i, v_{i+1}, h_{i+1}, w_{i+1})$ for $i = 2, \ldots, t-1$	$\frac{t-1}{2}$ cycles of length 6
	$(h_1, u_1, u_2, \ldots, u_t, v_1)$	a cycle of length $t+2$
$J(t) - v_1$	$(w_i, h_i, v_i, v_{i+1}, h_{i+1}, w_{i+1})$ for $i = 4, \ldots, t-1$	$\frac{t-3}{2}$ cycles of length 6
	$(v_1, h_1, w_1, w_2, w_3, h_3, v_3, v_2, h_2, u_2, u_3, \ldots, u_t)$	a cycle of length $t+8$

Hence, we have obtained that, for all of these graphs, there is always a 2–factor containing an even cycle. Thus, Definition 3.1(i) does not hold. Hence, $J(t)$ has no bold–edges.   □

Regarding gadget–pairs, we have computed so far, only the gadget–pairs in the Petersen graph P_{10} (cf. Lemma 3.5).

Lemma 5.2 *Let the Flower snark $J(t)$, for odd $t \geq 5$, the Blanuša2 snark P_{18}, P_{26}, and P_{34} defined as above. Then*

(i) P_{18}, P_{26} and P_{34} have no gadget–pairs;

(ii) The Flower snark $J(t)$ has no gadget–pairs.

Proof. For each of these graphs, we will verify that Definition 3.3(i) or (iv) does not hold.

(i) Fix the labelling on P_{18}, P_{26} and P_{34} as in Figures 4, 5 and 6. For these graphs, we have implemented a program, with the software package MAGMA,

in which we compute the edge–orbits under the action of the automorphism group; we consider all independent edges $g = cd$ from a chosen representative $f = ab$ of each edge–orbit and then we find all 2–factors of $G - \{f, g\}$. If any such 2–factors exist then condition 3.3(i) does not hold. Otherwise, we choose one of the edges $\{ac, ad, bc, bd\}$ (cf. Definition 3.3(iv)), say ac, then we compute all 2–factors of $G - \{f, g\} + \{ac\}$ and, in each case, we find a 2–factor for which condition 3.3(iv) does not hold.

We obtain the following results:

(a) In the graph P_{18}, for each representative $f = ab$ of one of the six edge–orbits, there are 22 possible independent edges $g = cd$. For most pairs there exists a 2–factor of $G - \{f, g\}$. Thus, condition 3.3(i) does not hold. The pairs f, g for which $G - \{f, g\}$ has no 2–factors are:

f	$(1,2)$	$(9,1)$	$(2,6)$		$(12,13)$			
g	$(7,8)$	$(12,13)$	$(4,8)$	$(12,13)$	$(4,8)$	$(11,3)$	$(15,5)$	$(17,7)$

For each of these pairs of edges, $G - \{f, g\} + \{ac\}$ admits a 2–factor F in which the cycle C_{ac} has odd length, or F has other even cycles besides C_{ac}, contradicting 3.3(iv). Hence, P_{18} has no gadget–pairs, since Definition 3.3(i) or (iv) does not hold.

(b) Similarly, in the graph P_{26}, for each representative $f = ab$ of one of the eight edge–orbits, there are 34 possible independent edges $g = cd$. For most pairs there exists a 2–factor of $G - \{f, g\}$. Thus, condition 3.3(i) does not hold. The pairs f, g for which $G - \{f, g\}$ has no 2–factors are:

f	$(2,5)$		$(7,8)$	$(7,12)$	$(10,11)$			
g	$(10,11)$	$(20,21)$	$(13,14)$	$(9,16)$	$(1,7)$	$(3,13)$	$(4,9)$	$(6,15)$	$(20,21)$

As for P_{18}, for each of these pairs of edges, $G - \{f, g\} + \{ac\}$ admits a 2–factor F in which the cycle C_{ac} has odd length, or F has other even cycles besides C_{ac}, contradicting 3.3(iv). Hence, P_{26} has no gadget–pairs, since Definition 3.3(i) or (iv) does not hold.

(c) Finally, in the graph P_{34}, for each representative $f = ab$ of one of the four edge–orbits, there are 46 possible independent edges $g = cd$. For most pairs there exists a 2–factor of $G - \{f, g\}$. Thus, condition 3.3(i) does not hold. The pairs f, g for which $G - \{f, g\}$ has no 2–factors are:

f	$(5,6)$	$(8,9)$					
g	$(11,12)$	$(1,5)$	$(2,7)$	$(3,13)$	$(4,11)$	$(18,19)$	$(28,29)$

As for P_{26}, for each of these pairs of edges, $G - \{f, g\} + \{ac\}$ admits a 2–factor F in which the cycle C_{ac} has odd length, or F has other even

cycles besides C_{ac}, contradicting 3.3(iv). Hence, P_{26} has no gadget–pairs, since Definition 3.3(i) or (iv) does not hold.

This concludes part (i) of the proof.

(ii) For the graphs $J(t)$, $t \geq 5$ odd, fix the labelling on the vertices of $J(t)$ as defined in Section 2.

Recall that in a cubic graph G, a 2–factor, F, determines a corresponding 1–factor, namely $E(G)-F$. In studying 2–factors in $J(t)$ it is more convenient to consider the structure of 1–factors.

If L is a 1–factor of $J(t)$ each of the t links of $J(t)$ contain precisely one edge from L. This follows from the argument in [1, Lemma 4.7]. Then, a 1–factor L may be completely specified by the ordered t–tuple $(a_1, a_2, \ldots, a_t)$ where $a_i \in \{u_i, v_i, w_i\}$ for each $i = 1, 2, \ldots, t$ and indicates which edge in L_i belongs to L. Together these edges leave a unique spoke in each IC_i to cover its hub. Note that $a_i \neq a_{i+1}$, $i = 1, 2, \ldots, t$ (i.e. they lie in different channels, for example if $a_i = u_i$, then $a_{i+1} \neq u_{i+1}$). To read off the corresponding 2–factor F simply start at a vertex in a base cycle at the first interchange. If the corresponding channel to the next interchange is not banned by L, proceed along the channel to the next interchange. If the channel is banned, proceed via a spoke to the hub (this spoke cannot be in L) and then along the remaining unbanned spoke and continue along the now unbanned channel ahead. Continue until reaching a vertex already encountered, so completing a cycle C_1. At each interchange C_1 contains either 1 or 3 vertices. Furthermore as C_1 is constructed iteratively, the cycle C_1 is only completed when the first interchange is revisited. Since C_1 uses either 1 or 3 vertices from IC_1 it can revisit either once or twice. If C_1 revisits twice then C_1 is a hamiltonian cycle which is not the case. Hence it follows that F consists of two cycles C_1 and C_2.

Let f, g be independent edges in $J(t)$. Since each of the t links of $J(t)$ contain precisely one edge from any given 1–factor L of $J(t)$, each 2–factor of $J(t)$ must contain exactly two edges of each link L_i. Therefore, if $f, g \in L_i$, for some $i \in \{1, \ldots, t\}$, then there is no 2–factor of $J(t)$ avoiding both, i.e. Definition 3.3(i) holds. Hence, to prove statement (ii) in this case, we need to verify that Definition 3.3(iv) does not hold. We need first to prove that for all other independent pairs that Definition 3.3(i) does not hold, namely $J(t) - \{f, g\}$ contains 2–factors. We need to consider the following four cases:

CASE 1: f, g are both hubs in different links.

Suppose $f = b_i b_{i+1}$ and $g = c_j c_{j+1}$, for $i \neq j$. Choose any 1–factor $L := (a_1, a_2, \ldots, a_t)$ such that $a_i = b_i$ and $a_j = c_j$. In the case $j = i+1$, $b_{i+1} \neq c_j$, since f and g are independent.

CASE 2: f, g are in a hub and a spoke with the same index.

Suppose $f = h_i b_i$ and $g = c_i c_{i+1}$. Choose a 1–factor $L := (a_1, a_2, \ldots, a_t)$

such that $a_{i-1} \neq b_{i-1}$ and $a_i = b_i$. Since f and g are independent, then $c_i \neq b_i$.

CASE 3: f, g are in a hub and a spoke with different indices.

Suppose $f = h_i b_i$ and $g = c_j c_{j+1}$, for $i \neq j$. Choose any 1–factor $L := (a_1, a_2, \ldots, a_t)$ such that $a_{i-1} \neq b_{i-1}$, $a_i \neq b_i$, $b_{i-1} \neq b_i$ (in different channels) and $a_j = b_j$. Moreover, if $j = i + 1$, choose $b_j \neq c_i$ (in different channels), which can always be done since there are three choices at each link by definition of $J(t)$.

CASE 4: f, g are both spokes in different interchanges.

Note that f, g cannot be two spokes in the same interchange since they are independent edges. Suppose $f = h_i b_i$ and $g = h_j c_j$, for $i \neq j$. Choose any 1–factor $L := (a_1, a_2, \ldots, a_t)$ such that $a_{i-1} \neq b_{i-1}$, $a_i \neq b_i$, $a_{j-1} \neq c_{j-1}$ and $a_j \neq c_j$, which is always possible since there are three choices at each link by definition of $J(t)$.

In each case, the 2–factor F corresponding to $J(t) - L$ is well defined and it avoids both f and g, thus Definition 3.3(i) does not hold.

This leaves us to prove that in the case $f, g \in L_i$ for some $i \in \{1, \ldots, t\}$, in which Definition 3.3(i) holds, Definition 3.3(iv) does not hold. To this purpose, we choose one of the edges $\{ac, ad, bc, bd\}$ (cf. Definition 3.3(iv)), say ac, and find a 2–factor for which Definition 3.3(iv) does not hold. Suppose, that $f = ab = a_i a_{i+1}$ and $g = cd = c_i c_{i+1}$, with $a_i, c_i \in \{u_i, v_i, w_i\}$, and consider the graph $J(t) - \{f, g\} + \{ac\} = J(t) - \{f, g\} + \{a_i c_i\}$.

Recall that the flower snark has the dihedral group D_{2t} as automorphism group ([13]) with vertex orbits $[h_1] := \{h_i : i = 1, \ldots, t\}$, $[w_1] := \{w_i : i = 1, \ldots, t\}$, and $[u_1] := \{u_i, v_i : i = 1, \ldots, t\}$. Then, w.l.o.g. we can consider the following two cases:

CASE A: $a_i, c_i \in [u_1]$, say $a_i = u_1$ and $c_i = v_1$.

In this case the graph $J(t) - \{f, g\} + \{u_1 v_1\}$ admits a 2–factor F of type $[3, t, 6, \ldots, 6]$ with cycles $(u_1 h_1 v_1)$, $(w_1 w_2 \ldots w_t)$, and $(u_i h_i v_i v_{i+1} h_{i+1} u_{i+1})$, for $i = 2, 4, \ldots, t - 1$.

CASE B: $a_i \in [u_1]$ and $c_i \in [w_1]$, say $a_i = u_1$ and $c_i = w_1$ respectively.

In this case the graph $J(t) - \{f, g\} + u_1 w_1$ admits a 2–factor F of type $[3, t + 6, 6, \ldots, 6]$ with cycles $(u_1 h_1 w_1)$, $(v_1 v_2 \ldots v_t h_t w_t w_{t-1} h_{t-1} u_{t-1} u_t)$, and $(u_i h_i w_i w_{i+1} h_{i+1} u_{i+1})$, for $i = 2, 4, \ldots, t - 3$.

In both cases, the cycle of F containing $a_i c_i$ is odd (of length 3) and it has some even cycles as well, implying that Definition 3.3(iv) does not hold.

Therefore, we can conclude that $J(t)$ has no gadget–pairs. $\square$

The results obtained so far give rise to the following partial characterization:

Theorem 5.3 *Let G be an odd 2–factored snark of cyclic edge–connectivity four that can be constructed from the Petersen graph and the Flower snarks*

using the bold–gadget dot product construction. Then $G \in \{P_{18},\ P_{26},\ P_{34}\}$.

Proof.   There is no possibility to construct other odd 2–factored snarks from the Flower snarks $J(t)$, $t \geq 5$ odd, with the bold–gadget dot product construction by Lemma 5.1 and Lemma 5.2(*ii*).

The Blanuša2 snark P_{18}, P_{26}, and P_{34} have been constructed iteratively via the bold–gadget dot product from the Petersen graph using the existence of bold–edges in Petersen (Lemma 3.2), P_{18} (Lemma 3.8), P_{26} (Proposition 4.1) and gadget–pairs in Petersen (Lemma 3.5). Since P_{34} has no bold–edges by Proposition 4.2 and P_{18}, P_{26}, P_{34} have no gadget–pairs by Lemma 5.2(*i*), there is no possibility to apply the bold–gadget dot product any further to these graphs.   □

Conjecture 5.4 *Let G be a cyclically 5–edge connected odd 2–factored snark. Then G is either the Petersen graph or the Flower snark $J(t)$, for odd $t \geq 5$.*

Remark 5.5 (*i*) *A minimal counterexample to Conjecture 5.4 must be a cyclically 5–edge connected snark of order at least 36 (cf. Remark 4.3). Moreover, as highlighted in [9], order 34 is a turning point for several properties of snarks.*
(*ii*) *It is very likely that, if such counterexample exists, it will arise from the superposition applied to one of the known odd 2–factored snarks.*
(*iii*) *We have also checked that the snark of order 46, of perfect matching index $\tau(G) = 5$, constructed by Hägglund in [22], counterexample to a strengthening of Fulkerson's conjecture [14, 30], is not odd 2–factored. Moreover, the Flower snark is odd 2–factored but it has $\tau(G) = 4$ (cf. [14]). Hence, there is no relation between odd 2–factored snarks and their perfect matching index being 5.*

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